Guaranteed Annuity Options
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Abstract

Under a guaranteed annuity option an insurer guarantees to convert a policyholder’s accumulated funds to a life annuity at a fixed rate. If the annuity rates provided under the guarantee are more beneficial to the policyholder than the prevailing rates in the market the insurer has to make up the difference. These guarantees are common in many US tax sheltered insurance products. They very were popular in the UK in the 1970’s and 1980’s when long term interest rates were high. At that time, the options were very far out of the money and insurance companies apparently assumed that interest rates would remain high and thus that the guarantees would never become active. In the 1990’s, as long term interest rates began to fall the value of these guarantees rose. Two other factors added to the cost of these guarantees. First, strong stock market performance meant that the amounts to which the guaranteed applied increased significantly. Second, the mortality assumption implicit in the guarantee did not anticipate the improvement in mortality which took place. The emerging liabilities under these guarantees threatened the solvency of some companies and lead to the closure of Equitable Life (UK) to new business. In this paper we explore the pricing and risk management of these guarantees.
1 Introduction

Insurance companies often include guarantees in their products. These guarantees provide options to their policyholders which in some circumstances can be valuable to their customers. In the past some of these options have been viewed by insurers as having little or zero value and were ignored when setting up reserves. However many of these options were very long dated lasting 30 to 40 years and over this time span there can be significant fluctuations in economic variables which affect the value of these options. The case of guaranteed annuity options in the UK provides a dramatic illustration of this phenomenon and is the subject of our paper.

Guaranteed annuity options have proved to be a significant risk management challenge for several UK insurance companies. Bolton et al (1997) describe the origin and nature of these guarantees. They also discuss the factors which caused the liabilities associated with these guarantees to increase so dramatically in recent years. These factors include a decline in long term interest rates and improvements in mortality. For many contracts the liability is also related to equity performance and in the UK common stocks performed very well during the last two decades of the twentieth century.

We now describe these guarantees and explain why they became such a severe problem for the UK insurance industry. Under a guaranteed annuity the insurance company guarantees to convert the maturing policy proceeds into a life annuity at a fixed rate. Typically, these policies mature when the policyholder reaches a certain age. In the UK a popular guaranteed rate for males, aged sixty five, was 111 per annum per 1000 and we use this rate in our illustrations. If the prevailing annuity rates at maturity provide an annual payment that exceeds 111, a rational policyholder would opt for the prevailing market rate. On the other hand, if the prevailing annuity rates at maturity produce a lower amount than 111, a rational policyholder would take the guaranteed annuity rate. As interest rates rise the annuity amount purchased by a lump sum of 1000 increases and as interest rates fall the annuity amount available per 1000 falls. Hence the guarantee corresponds to a put option on interest rates.

These guarantees began to be included in some UK policies in the 1950’s and they became very popular in the 1970’s and 1980’s. The inclusion of these guarantees was discontinued by the end of the 1980’s but given the long term nature of this business these they still affect a significant number of contracts. Long term interest rates in many countries were quite high in 1970’s and
1980’s and the UK was no exception. During these two decades the average UK long term interest rate was around 11%. The interest rate implicit in the guaranteed annuity options depends on the mortality assumption but based on the mortality basis used in the original calculations the break-even interest rate was in the region of $5 - 6$ percent. When these options were granted, they were very far out of the money and the insurance companies apparently assumed that interest rates would never fall to these low levels again and thus that the guarantees would never become active. As we now know this presumption was incorrect and interest rates did fall in the 1990’s.

The guaranteed annuity conversion rate is a function of the assumed interest rate and the assumed mortality rate$^1$. However there was a dramatic improvement in the mortality of the class of lives on which these guarantees were written during the period 1970-2000. This improvement meant that the break-even interest rate at which the guarantee kicked in rose. This point can be illustrated using term certain annuities of increasing length. A lump sum of 1000 is equivalent to a thirteen year annuity certain of 111 p.a. at 5.70%. However if we extend the term of the annuity to sixteen years the interest rate rises to 7.72%. Hence if mortality rates improve so that policyholders live longer, the interest rate at which the guarantee becomes effective, will increase.

Using standard actuarial notation, the value of the guarantee at maturity(time $T$) for the benchmark contract is

$$S(T) \max \left[ \left( \frac{a_{65}(T)}{9} - 1 \right), \ 0 \right]$$

(1)

where $S(T)$ is the amount of the proceeds at time $T$ and $a_{65}(T)$ is market annuity rate at time $T$ for a life aged 65. The market annuity rate will depend on the prevailing long term interest rates, the mortality assumptions used and the expense assumption. We will ignore expenses and use the current long term government bond yield as a proxy for the interest rate assumption. We see that the option will have a positive value at maturity (be in the money) whenever the current annuity factor exceeds the guaranteed factor(9 in this case).

We see from equation (1)that for a maturing policy the size of the option liability, if the guarantee is operative, will be proportional to $S(T)$:

$^1$Bolton et al note that when many of these guarantees were written it was considered appropriate to use a mortality table with no explicit allowance for future improvement such as $a(55)$. 

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the amount of proceeds to which the guarantee applies. The size of $S(T)$ will depend on the nature of the contract and also on the investment returns attributed to the policy. The way in which the investment returns are determined depends on the policy contract. Under a traditional UK with profits contract profits are assigned using reversionary bonuses and terminal bonuses. Reversionary bonuses are assigned on a regular basis as guaranteed additions to the basic maturity value and are not distributed until maturity. Terminal bonuses are declared when the policy matures to reflect the investment experience over the term of the contract.

The size of the reversionary bonuses depends both on the investment performance of the underlying investments and the smoothing convention used in setting the bonus level. The terminal bonus is not guaranteed but during periods of good investment performance it can be quite significant sometimes of the same size as the basic sum assured. Bolton et al (1997) estimate that with profits policies account for eighty percent of the total liabilities for contracts which include a guaranteed annuity option. The remaining contracts which incorporate a guaranteed annuity option were mostly unit lined policies.

We now briefly describe unit linked contracts. In contrast to with profits contracts, the investment gains and losses under a unit linked (equity linked) contract are distributed directly to the policyholder’s account. Contracts of this nature are more transparent than with profits policies and they have become very popular in many countries in recent years. Under a unit linked contract the size of the option liability, if the guarantee is operative, will depend directly on the investment performance of the assets in which the funds are invested. In the UK there is a strong tradition of investing in equities and during the twenty year period from 1980 until 2000 the rate of growth on the major UK stock market index was a staggering 18% per annum.

We see that three principal factors contributed to the growth of the guaranteed annuity option liabilities in the UK over the last few decades. First, there was a large decline in long term interest rates over the period. Second, there was a significant improvement in longevity that was not factored into the initial actuarial calculations. Third, the strong equity performance during the period served to further increase the magnitude of the liabilities. It would appear that these events were not considered when the guarantees were initially granted. The responsibility for long term financial solvency of insurance companies rests with the actuarial profession. It will be instruc-
tive to examine what possible risk management strategies could have been or should have been employed to deal with this situation. It is clear now with the benefit of hindsight that it was imprudent to grant such long term open ended guarantees of this type.

There are three main methods of dealing with the type of risks associated with writing financial guarantees. First, there is the traditional actuarial reserving method whereby the insurer sets aside additional capital to ensure that the liabilities under the guarantee will be covered with a high probability. The liabilities are estimated using a stochastic simulation approach. The basic idea is to simulate the future using a stochastic model of investment returns. These simulations can be used to estimate the distribution of the cost of the guarantee. From this distribution one can compute the amount of initial reserve so that the provision will be adequate say 99% of the time. The second approach is to reinsure the liability with another financial institution such as a reinsurance company or an investment bank. In this case the insurance company pays a fee to the financial institution and in return the institution agrees to meet the liability under the guarantee. The third approach is for the insurance company to set up a replicating portfolio of traded securities and adjust (or dynamically hedge) this portfolio over time so that at maturity the market value of the portfolio corresponds to the liability under the guaranteed annuity option.

Implementations of these three different risk management strategies have been described in the literature. Yang, Waters and Wilkie(2002) describe the actuarial approach based on the Wilkie model. Dunbar(1999) describes an example of the the second approach. The insurance company, Scottish Widows offset its guaranteed annuity liabilities by purchasing a structured product from Morgan Stanley. Pelsser(2002) analyzes a hedging strategy based on the purchase of long dated receiver swaptions.

In this paper we will discuss a number of the issues surrounding the valuation and risk management of these guarantees. We will also discuss the degree to which different risk management approaches would have been possible from 1980 onwards.

The layout of the rest of the paper is as follows. Section Two provides background detail on the guaranteed annuity options and the relevant institutional framework. We examine the evolution of the economic and de-

\footnote{One such model, the Wilkie Model was available in in the UK actuarial literature as early as 1980.}
mographic variables which affect the value of the guarantee. In particular we provide a time series of the values of the guarantee at a maturity for a representative contract. In Section Three we use an option pricing approach to obtain the market price of the guarantee. Section Four documents the time series of market values of the guarantee Although the model is a simple one we argue that it does a reasonable job of estimating the market value of the option.

One suggestion for dealing with these guarantees involves the insurer purchasing long dated receiver swaptions. We describe this approach in Section Five. We conclude that buying swaptions provides an efficient solution for dealing with the interest rate risk dimension but it cannot deal effectively with the equity risk and the mortality risk. Section Six examines a number of the conceptual and practical issues involved in dynamic hedging the interest rate risk. In Section Seven we discuss the issues involved in hedging the equity risk and the mortality risk. Section Eight discusses the risk management solutions that could have been applied and comments on the lessons to be learned from this episode.

2 Maturity Value of the Guarantee

In this section we document the evolution of the emerging liability under the guaranteed annuity option. Specifically we examine the magnitude of the guarantee for a newly maturing policy over the last two decades. In these calculations the policy proceeds are held constant at 100. We assume, to start with, that the mortality basis used to compute the market rate corresponds to that used in the original calculations. Of course we now know that mortality improved over the period of interest but this will give us a benchmark. On the basis of the so-called a(55) mortality the break even-interest rate for a life annuity\(^3\) is 5.61%. On this mortality basis a lump sum of 1000 will purchase

\[^3\text{We assume the annuity is payable annually in arrear and has a five year guarantee period. The precise level of the break even interest rate will depend on the features of the annuity contract. These include the frequency of payment, i.e. whether it is payable monthly or yearly and also whether the annuity is assumed to be payable in advance or arrears. Bolton et al(1997) provide extensive tables of the break even interest rates for different types of annuities and different mortality tables. They assume a two percent initial expense charge which we do not include. Thus in their Table 3.4 the value for the break even interest rate for a male aged 65 for an annuity of 111 payable annually in arrear with a five year guarantee is 5.9%. This is consistent with our figure of 5.6 \% when we}\]
an annuity of 111. If current annuity rates are calculated using this mortality assumption the guarantee will not be in the money if long term interest rates are greater than 5.61%.

Figure 1 illustrates the behavior of long term interest rates in the UK since 1970. We note that rates remained quite high for the period 1970-1990 and started to decline in the 1990’s. There was a large dip in long rates at the end of 1993 and long rates first fell below 6% in 1998 and have hovered in the 4 – 6 range until the present\textsuperscript{4} time.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{long_rates.png}
\caption{Long UK interest rates 1970-2002}
\end{figure}

The maturity value of the guaranteed option will depend on the long term interest rate at the policy maturity. If this market rate is 5% per

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\textsuperscript{4}Late in the fall of 2002 at the time of writing
Figure 2: Interest rates levels that will trigger the guarantee. Bottom horizontal line (5.6) based on a(55) mortality. Middle horizontal line (7.3) based on PMA80(C10) mortality. Top horizontal line (8.2) based on PMA92(C20) mortality.
annum the corresponding annuity rate, based on the $a(55)$ table, is 9.40. If the policyholder buys an annuity on the open market he receives\(^5\) 106.38 per annum per 1000 of proceeds. Since the guaranteed amount is 111 it is better for the policyholder to take the guaranteed amount. The value of the guarantee corresponds to an annuity of 4.62 ($= 111 - 106.38$) per annum which is equivalent to a lump sum of 43.4 ($= 9.4 \times 4.68$). Thus the additional liability arising from the existence of the guarantee is 43.4 per thousand or 4.34 per hundred.

During the period 1970-2000 there was a very significant improvement in the mortality of UK males especially at the older ages. We can illustrate this improvement by using three\(^6\) mortality tables capture the full extent of this improvement in UK annuitant mortality over the thirty year period 1970-2000. For the period 1970-1980 we use the $a(55)$ mortality table. For the second decade from 1980 until 1990 we use PMA80(C10) mortality and for third decade from 1990 until 2000 we use PMA92(C20) mortality. The increase in longevity is quite dramatic over this period. The expectation of life for a male aged 65 under the $a(55)$ mortality is 14.3 years, 16.9 years under PMA80(C10) table and 19.8 years under the PMA92(C20) table. Thus the expected future lifetime of males aged 65 increased by over five years from the $a(55)$ table to the PMA92(C20) table.

As a consequence of the mortality improvement the cost of immediate annuities increased significantly over this period independently of the impact of falling interest rates. The break-even interest rate for our benchmark contract under the $a(55)$ table is 5.6. Under the PMA80(C10) table it is 7.3\% and under the PMA92(C20) table it is 8.2\%. This increases in the level of the break even interest rate has profound implications for the cost of a maturing guaranteed annuity option. For example, if the long term market rate of interest is 5\%, the value of the option for a maturing policy based on PMA80(C10) is 164.1 and the corresponding value based on PMA92(C20) is 293.9. Note that we do not need any type of option formula to perform these calculations. Figures 3, 4 and 5 show the magnitude of the option liability for our benchmark contract under the three mortality assumptions. There is no liability on maturing contracts until the 1990’s. Also not that the mortality

\(^5\)Note that $\frac{1000}{9.4} = 106.38$.

\(^6\)The $a(55)$ table was being used to compute annuity values in the 1970’s. The PMA80(C10) table is based on UK experience for the period 1979-1982 and projected to 2010 to reflect mortality improvements. The PMA92(C20) table is based on UK experience for the period 1991-1994 and projected to 2020 to reflect mortality improvements.
assumption has a profound impact on the size of the liability.

Figure 3: Value of maturing guarantee based on a(55) mortality. Threshold rate 5.6%.
Figure 4: Value of maturing guarantee based on PMA80(C10) mortality. Threshold rate 7.3%.
<table>
<thead>
<tr>
<th>Year</th>
<th>Option Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1980</td>
<td>0</td>
</tr>
<tr>
<td>1985</td>
<td>5</td>
</tr>
<tr>
<td>1990</td>
<td>10</td>
</tr>
<tr>
<td>1995</td>
<td>15</td>
</tr>
<tr>
<td>2000</td>
<td>20</td>
</tr>
<tr>
<td>2005</td>
<td>25</td>
</tr>
</tbody>
</table>

Figure 5: Value of maturing guarantee based on PMA92(C20) mortality. Threshold rate 8.2%.
We have already noted that during the period 1980-2000 UK equities performed extremely well. This good equity performance resulted in increased levels of bonus to the with profits polices. For many contracts this meant that the volume of proceeds to which the guarantee applied also increased thereby increasing the liability under the guarantee. In the case of unit linked polices the gains are passed directly to the policyholder, apart from the various expenses. If we assume that a unit linked contract earned the market rate of 18% minus 300 basis points this still leaves a return of 15%. At this growth rate an initial single premium of 100 will accumulate to 1636.7 after twenty years. This growth would be directly reflected in the value of the guarantee.

To summarize, we have discussed the evolution of the value of the liability for a sequence of maturing contracts. This analysis indicates how the three factors:-
- The fall in long term interest rates
- The improvement in mortality
- The strong equity performance

-served to increase the cost of the guarantee. Note that this evaluation does not require any stochastic analysis. We simply computed the value of the option at maturity. We next discuss the evaluation of these options prior to maturity.

### 3 Derivation of Option Formula

We will develop the formula for the option in steps. First we deal with simpler contracts which involve only interest rate risk. Then we introduce mortality risk as well. Finally we derive a formula for the guaranteed annuity option.

We will use the fact that the guaranteed annuity option is similar to a call option on a bond where the coupon payments correspond to the guaranteed annuity payments. We will first assume that the mortality risk is independent of the financial risk and that diversifiable. We base the valuation of the option on the one factor Vasicek(1977) model. This model assume that the short term interest rate follows a mean reverting Ornstein Uhlenbeck process, and admits simple analytical solutions for bond prices and the prices of options on zero coupon bonds. In 1989, Farshid Jamshidian published a paper in the Journal of Finance in which he derived a simple formulae for the price of an option on a coupon paying bond as a linear combination of options on zero coupon bonds. However neither the original Vasicek model nor the Jamshidian model reproduced the market prices of the current term structure of current term structure these models pure discount bonds. Phil Dybvig(1989) showed that how to adjust a one factor stochastic interest rate model so that it could reproduce the market term structure.

We start by valuing a very basic contract. We assume that the market price, at time \( t \), of a zero coupon bond that will pay one unit at time \( s \geq t \) is \( D(t, s) \). We assume an arbitrage free financial market and we also assume that there is a complete spectrum of bond maturities. To begin with, the only random variable is interest rate risk. If we are now at time \( t \) we know the prices of all the zero coupon bonds with maturity \( s \geq t \). We now consider
a contract that pays one unit at times \((T+j)\), where \(j = 1, 2, \ldots, k\) and \(T > t\). Note that the market value of this contract at current time \(t\) is

\[
V(t) = \sum_{j=1}^{k} D(t, T+j)
\]

This result follows from the no arbitrage assumption.

We can also express the current value of this payment stream as follows

\[
\frac{V(t)}{D(t, T)} = E_{Q_T} \left[ \frac{V(T)}{D(T, T)} \bigg| t \right] = E_{Q_T} [V(T) \mid t ]
\]

This is because in the absence of arbitrage the prices deflated by a suitable numeraire are martingales. We can use any traded asset\(^7\) as numeraire. Here we use the zero coupon bond which matures at time \(T\) as the numeraire. We denote the associated probability measure by the symbol \(Q_T\). Equation (2) provides a valuation formula for any payoff \(V(T)\) and we use it extensively in this section.

The expected value, under \(Q_T\), of any pure discount bond with maturity \((\geq T)\) can be readily obtained using the same valuation formula.

\[
E_{Q_T} \left[ D(T, T+j) \mid t \right] = \frac{D(t, T+j)}{D(t, T)}
\]

The ratio on the right hand side is often called the time \(T\) forward price at \(t\), of the pure discount bond with maturity \((T+j)\).

We now introduce the mortality random variable. We will deal with contracts where the payments are contingent upon the survival of a given life. Under an immediate annuity the life receives one unit per annum as long as he or she survives. The actuarial present value at \(T\) of an immediate annuity to a life aged \(R\) at \(T\), is

\[
a_T(R) = \sum_{j=1}^{J} j p_R D(T, T+j)
\]

where \(j p_R\) represents the probability that the life aged \(R\) will survive for a further \(j\) years. The limiting age of the mortality table is denoted by \(\omega\) and we set \(J = (\omega - R)\).

\(^7\)whose price is always positive
This actuarial present value corresponds to the expectation over the distribution of the (curtate) future lifetime of the life in question. Let \( \tau \) denote the future lifetime of a life aged \( R \) at time \( T \). Consider the random variable

\[
Y(\tau | T) = \sum_{j=1}^{\tau} D(T, T + j).
\]

This corresponds to the market value of an annuity certain payable for \( \tau \) years. Note that \( D(T, T + j) \) is known at time \( T \) for all \( j \). The probability that \( \tau \in (k-1, k] \) is

\[
k-1p_R q_{R+k-1}.
\]

Using these probabilities, the expected value of the annuity certain payable for the random future lifetime is

\[
E_{P_S} [Y(\tau | T)] = \sum_{k=1}^{J} k-1p_R q_{R+k-1} \left[ Y(\tau) | T \right].
\]  \hspace{1cm} (4)

The expectation here is taken with respect to the survival probabilities, \( P_S \).

It is easy to show that this expectation can be converted to the expression for \( a_R(T) \) on the right hand side of equation(3).

Our next task is to derive an expression for the market value, at time \( t \), of a deferred annuity. We assume the life in question is aged \( x \) at current time \( t \). At time \( T > t \), this individual will be aged \( R = x + (T - t) \) assuming survival. At current time \( t \leq T \), these future payments are random variables: both with respect to mortality and also with respect to interest rates. Milevsky and Promislow (2002) discuss the valuation of insurance contracts allowing for both sources of randomness. The results are simpler if we assume that the force of mortality\(^8\) is deterministic and for now we will make this assumption. Under this assumption the interest rate risk is independent of the mortality risk.

There is an important implication of this mortality assumption. Assume we have a life aged \( x \) at current time \( t \). At time \( T > t \) this life will either survive and reach age \( R = x + T - t \) or die in the interval \((t, T)\). Our mortality assumption implies that we know, at current time \( t \), the distribution of the future lifetime of the life conditional on reaching age \( R \). In other words we

\(^8\)The force of mortality corresponds to the hazard rate in modelling default risk (see Duffie and Singleton(1997)).
can accurately predict at time $t$ the force of mortality that will operate during $[T, T + J)$. Milevsky and Promislow provide a clear discussion of this point.

Let $V(t)$ be the market value at time $t$ of the deferred annuity that starts at time $T$. We have

$$
\frac{V(t)}{D(t, T)} = E \left[ (Y(\tau_x) 1_{\tau_x > (T-t)}) \mid t \right]
$$

where the expectation is taken over the joint distribution of $Q_T$ and $P_S$ and $\tau_x$ is the future lifetime of a life aged $x$ at time $t$. Because the mortality risk is assumed to be diversifiable and independent of the interest rate risk we can write

$$
\frac{V(t)}{D(t, T)} = (T-t)p_x \sum_{j=1}^{J} p_R \frac{D(t, T+j)}{D(t, T)}
$$

Hence we have

$$
V(t) = (T-t)p_x \sum_{j=1}^{J} p_R D(t, T+j)
$$

Note that the deferred annuity can be expressed as a linear combination of zero coupon bonds.

We now turn to business at hand: the valuation of the guaranteed annuity option. Let $G(T)$ denote the value of this option at maturity. We have

$$
G(T) = \frac{S(T) (a_{R(T)} - g)^+}{g} 1_{\tau_x > T_d}
$$

where $T_d = T - t$ and $g$ is the guaranteed annuity conversion rate. In our benchmark example factor $g = 9$. Proceeding as before the value of the option at time $t$ is given by

$$
\frac{G(t)}{D(t, T)} = (T-t)p_x E_{Q_T} \left[ \frac{S(T) (a_{R(T)} - g)^+}{g} \mid \tau_x > T_d \right]
$$

If we assume further that $S(T)$ is independent of interest rates then we have
\( G(t) = \frac{\tau - \tau_p}{\tau} D(t, T) E_{Q_T} \left[ S(T) \right] E_{Q_T} \left[ (a_R(T) - g)^+ \mid \tau_x > T_d \right] \)
\[ = \frac{\tau - \tau_p}{\tau} \frac{S(t)}{g} E_{Q_T} \left[ (a_R(T) - g)^+ \mid \tau_x > T_d \right] \]

The last line follows because
\[
\frac{S(t)}{D(t, T)} = E_{Q_T} \left[ S(T) \mid t \right]
\]

Inserting the expression for \( a_R(t) \) from (3) we have
\[
E_{Q_T}[(a_R(T) - g)^+ \mid \tau_x > T_d)] = E_{Q_T} \left[ \sum_{j=1}^{J} j p_R D(T, T + j) - g \right] t
\]

The expression inside the expectation on the right hand side corresponds to a call option on a coupon paying bond where the coupon payments correspond to the survival probabilities. Jamshidian(1989) cleverly noted that in the case of a one factor interest rate model this option on a portfolio of zero coupon bonds could be expressed as a portfolio of options on zero coupon bonds. Hence we now assume that the interest rate dynamics are generated by a single factor. Specifically we assume that the short interest rate follows the one factor Ornstein Uhlenbeck process as assumed by Vasicek. Let \( a_j = j p_R \) so that the coupon bond value at time \( T \), is
\[
\sum_{j=1}^{J} a_j D(T, T + j)
\]

Note that the market value at time \( t \) of this coupon bond is
\[
P(t) = \sum_{j=1}^{J} a_j D(t, T + j)
\]

With this notation our call option has a value at time \( T \) of
\[
(P(T) - g)^+
\]

Let \( r_T^* \) denote the value of the short rate for which
\[
\sum_{j=1}^{J} a_j D^*(T, T + j) = g
\]
where we use the asterisk to signify that each zero coupon bond is evaluated at \( r^*_T \). We now define new strike prices \( K_j \) as follows

\[ K_j = D^*(T, T + j) \]

Jamshidian proved that the market price of the option on the coupon bond with strike price \( g \) is equal to a portfolio of options on the individual zero coupon bonds with strike prices \( K_j \). Specifically we have

\[ C[P(t), g, t] = \sum_{j=1}^{J} a_j C[D(t, T + j), K_j, t] \]

where \( C[P(t), g, t] \) is the price at time \( t \) of a call option on the coupon bond with strike price \( g \) and \( C[D(t, T + j), K_j, t] \) is the price at time \( t \) of a call option on the zero coupon bond with maturity \((T + j)\) and strike price \( K_j \). For the Vasicek model these call prices have a simple Black Scholes type expression.

We can use Jamshidian’s result to obtain an explicit expression for \( G(t) \). Recall that

\[ G(t) = \frac{\tau - t p_x S(t)}{g} E_{Q_T}[(P(T) - g)^+ \mid t] \]

From the numeraire valuation equation we have

\[ \frac{C[P(t), g, t]}{D(t, T)} = E_{Q_T}[(P(T) - g)^+ \mid t] \]

Pulling all the pieces together we have

\[ G(t) = \frac{\tau - t p_x S(t)}{g} \sum_{j=1}^{J} a_j \frac{C[D(t, T + j), K_j, t]}{D(t, T)} \tag{7} \]

The explicit formula for each individual bond option under the Vasicek model is

\[ C[D(t, T + j), K_j, t] = D(t, T + j) N(h_1) - K_j D(t, T) N(h_2) \]

where

\[ h_1 = \frac{\log \frac{D(t, T + j)}{D(t, T) K_j}}{\sigma_p} + \frac{\sigma_p}{2}, \]
\[ h_2 = \frac{\log \frac{D(t, T + j)}{D(t, t)K_j}}{\sigma_p} = \frac{\sigma_p}{2}, \]

and

\[ \sigma_p = \sigma \sqrt{\frac{1 - e^{-2\kappa(T-t)}}{2\kappa}} \left(1 - e^{-\kappa j}\right). \]

The parameters \( \kappa, \theta \) and \( \sigma \) characterize the dynamics of the short rate of interest under the Vasicek process. The price of the zero coupon bond under this model when the short rate is \( r(t) \) is

\[ D(t, t + s) = \exp \left\{ -\theta \tau - (r(t) - \theta) \left( \frac{1 - e^{-\kappa s}}{\kappa} \right) + \frac{\sigma^2}{4 \kappa^3} \left( 4e^{-\kappa s} - e^{-2\kappa s} + 2\kappa s - 3 \right) \right\} \]  

(8)

So far we have assumed that the market price of the zero coupon bond is equal to the model price. In general, this will not be the case for the Vasicek model. We now describe Dybvig’s adjustment for calibrating the model to the prevailing market prices of zero coupon bonds. Denote the market price of the zero coupon bond maturing at time \( s \) by \( D_{\text{mar}}(t, s) \) and denote the model price of the zero coupon bond maturing at time \( s \) by \( D_{\text{mod}}(t, s) \). Define their ratio as \( Z(t, s) \).

\[ Z(t, s) = \frac{D_{\text{mar}}(t, s)}{D_{\text{mod}}(t, s)} \forall s \geq t. \]

When we use Dybvig’s approach the numeraire is the actual bond traded in the market which matures at time \( T \). Its price at time \( t \) is \( D_{\text{mar}}(t, T) \). When we use this bond as the numeraire we denote the associated probability measure by \( Q^*_T \). The current price of a security which has a payoff of \( V(T) \) at time \( T \), is given by

\[ \frac{V(t)}{D_{\text{mar}}(t, T)} = E_{Q^*_T} \left[ V(T) \mid t \right] \]  

(9)

As before we express the option on the coupon paying bond as a sum of options on zero coupon bonds. The adjusted strike price \( \kappa_j \) of the individual options is found as follows. Let \( r^*_T \) denote the value of the short rate for which

\[ \sum_{j=1}^{J} b_j D_{\text{mod}}^*(T, T + j) = g \]

9The model price under the one factor Vasicek model. Previously we used the notation \( D(t, s) \) for the model price
where
\[ b_j = a_j \left( \frac{Z(t, T + j)}{Z(t, T)} \right) \]
and the \( D^*_{mod}(T, T + j) \) are computed under the Vasicek model based on the short rate \( r_T \). We set
\[ \kappa_j = D^*_{mod}(T, T + j) \]

We can now state Dybvig’s result for the price of an option on a coupon bond where the bond prices are calibrated to the market and the interest rate dynamics are given by a one factor model. The market price at time \( t \) of the option on the coupon bond is given by
\[
C[P_{mar}(t), g, t] = Z(t, T) \sum_{j=1}^{J} b_j C[D_{mar}(t, T + j), \kappa_j, t],
\]
where
\[
P_{mar}(t) = \left( \sum_{j=1}^{J} a_j D_{mar}(t, T + j) \right)
\]
is the current market price of the coupon bond. We can use this to obtain the formula for the price of the guaranteed annuity option when bond prices are calibrated to market values. The guaranteed annuity option price is similar to equation (7) except that it now involves an expectation over \( Q_T^* \). However from the basic valuation equation (9)

\[
\frac{C[P_{mar}(t), g, t]}{D_{mar}(t, T)} = \frac{Z(t, T)}{D_{mar}(t, T)} \sum_{j=1}^{J} b_j C[D_{mar}(t, T + j), \kappa_j, t] = \frac{1}{D_{mod}(t, T)} \sum_{j=1}^{J} b_j C[D_{mod}(t, T + j), \kappa_j, t]
\]

Using this result the formula for the price of the guaranteed annuity option when we calibrate to the current term structure is given by
\[
G(t) = \frac{P_x(t)}{g} \sum_{j=1}^{J} b_j C[D_{mod}(t, T + j), \kappa_j, t] D(t, T)
\]

\[10\text{Note that the } b_j \text{ are known constants at time } t.\]
4 Valuation of Guaranteed Option

In this section we will derive a time series of market values for the guarantee based on the formula derived in the last section. The technology for pricing interest rate options was in its infancy in 1980 but by 1990 the models we will use were in the public domain. We will make the case that reasonable estimates of the market value of the guarantee can be derived from the one factor stochastic interest rate model. We will use the model to estimate the value of the guarantees for the period 1980-2002.

We showed in the previous section how one can use the Vasicek model in conjunction with the insights of Jamshidian and Dybvig to produce a one factor model that can be used to obtain a formula for the market price of the guaranteed annuity option. A similar formula has also been derived by Ballotta and Haberman(2002). They start from the Heath Jarrow Morton model and then restrict the volatility dynamics of the forward rate process to derive tractable formulae.

In the one factor Vasicek model the short interest, \( r(t) \) follows an Ornstein Uhlenbeck (OU) process under the risk neutral (or \( Q \)) measure

\[
dr(t) = \kappa(\theta - r(t))dt + \sigma_v dW_t
\]

(11)

where \( \kappa, \theta \) and \( \sigma_v \) are constants and \( W_t \) is a standard Brownian motion. This process is very tractable since the distribution of \( r(s), (s > t) \) is normal with mean

\[
E(r(s)|r(t)) = e^{-\kappa(s-t)}r(t) + (1 - e^{-\kappa(s-t)})\theta
\]

(12)

and variance

\[
Var(r(s)|r(t)) = \sigma_v^2 \frac{(1 - e^{-2\kappa(s-t)})}{2\kappa}
\]

(13)

We used the following parameter estimates to compute the market values of the guaranteed annuity option

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa )</td>
<td>0.35</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.08</td>
</tr>
<tr>
<td>( \sigma )</td>
<td>0.025</td>
</tr>
</tbody>
</table>
These parameters are broadly comparable with estimates that have been obtained in the literature based on UK data for roughly this time period. See Nowman (1997) and Yu and Phillips (2001). Because of the way we calibrate the model to the market term structure the option prices depend essentially on the volatility parameter.

Figure 7 illustrates how the market value of the option as percentage of the current fund value changes over time. We assume that the option has remaining time to maturity of ten years so that the age of the policyholder at the option valuation date is 55. We ignore the impact of lapses and expenses and we assume that all policyholders will take up the option at maturity if it is in their interest. Our first set of calculations are based on $a(55)$ mortality. The term structure at each date is obtained by assuming that the 2.5% consol yield operates for maturities of five years and longer and that the yields for maturities one to five are obtained by a linear interpolation between the Treasury Bill rate and the five year rate.
Figure 7: Market Values of guaranteed annuity option for benchmark contract based a(55). Interest rate parameters $\kappa = 0.35$, $\theta = 0.08$, $\sigma = 0.025$. 

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Figure 8: Market Values of guaranteed annuity option for benchmark contract based on contemporary mortality PMA92(C20). $\kappa = 0.35$, $\theta = 0.08$, $\sigma = .025$
Figure 9: Comparison of Market Values of guaranteed annuity option for benchmark contract based on three different volatility assumption. These are $\sigma = 0.015, 0.025, 0.035$. $\kappa = 0.35$, $\theta = 0.08$. Values based PMA92(C20) mortality

Figure 9 illustrates how the cost of the guaranteed annuity option for our benchmark contract varies with the volatility assumption. We used three different volatility assumptions: $\sigma = 0.015, 0.025, 0.035$. The market values are relatively insensitive to the volatility assumption for long periods. Indeed the only periods where we can distinguish the three separate curves corresponds to periods when the long term interest rate is close to the strike price of the option: 8.2%. We know from the basic Black Scholes equation that the sensitivity of an option to the volatility is highest when the underlying asset price is close to the strike price. If the option is well out of the money or deeply in the money the price of the option is relatively insensitive to the volatility assumption. This same intuition is at work here.
At first sight it is surprising that a simple one factor model can give good estimates of the market value of the option since actual interest rate dynamics are much too complicated to be represented by such a model. The reason is that we have calibrated the model to the input term structure so that it reproduces the market prices of all the zero coupon bonds. In addition the prices are quite robust to the volatility assumption for the realized market conditions. However we stress that such a simple model will not be adequate for hedging purposes. John Hull also notes The reality is that relatively simple one-factor models if used carefully usually give reasonable prices for instruments, but good hedging schemes must explicitly or implicitly assume many factors.

5 The Swaption Solution

Swaps have became enormously important financial instruments for managing interest rate risks. They are often more suitable than bonds for hedging interest rate risk since the swap market is more liquid. Furthermore while it can be difficult to short a bond, the same exposure can easily be arranged in the swap market by entering a so called payer swaption. Pelsser (2002) shows that long dated receiver swaptions are natural vehicles for dealing with the interest rate risk under guaranteed annuity options. These are options on interest rate swaps and were developed in the late 1980’s. In this section we give a brief description of interest rate swaps and swaptions and discuss how these instruments can be used to hedge the guaranteed annuity options. In this section we give some a brief description of interest rate swaps and swaptions and discuss how these instruments can be used to hedge the guaranteed annuity options.

A swap is an agreement between two parties (known as counterparties) to exchange periodic payments for a certain time period. In a generic interest rate swap one counterparty pays a fixed coupon rate to the other in exchange for the receipt of floating rate payments. Under a swap contract the party that pays fixed and receives floating is said to have entered a payer swaption. Conversely the other counterparty party that receives fixed and pays floating is said to have entered a receiver swaption. The floating rate payments are computed; with reference to some short term interest rate eg; three month LIBOR. The coupon payments are computed based on some fixed notional principal. There is no exchange of principal, only of coupon payments.
The fixed rate swap coupon rate in an interest rate swap can easily be computed since the swap coupon is arranged so that the swap is worth zero at inception. Suppose the notional principal is just one unit. We can view the swap as an arrangement to swap a fixed rate bond for a floating rate bond. These two bonds have the same maturity date and the same principal. Suppose the principal amount is one unit. In this case the current price of the floating rate bond is one unit. Assume the swap contract lasts for \( m \) periods and that the current time is \( t \). Assume \( T = t + m \). The current value of one unit payable after \( n \) periods is; \( D(t, T) \). Hence the value of the floating rate coupons is

\[
1 - D(t, T)
\]

If we denote the fixed rate coupon by \( sc(t, t + m) \), then the value of the fixed rate coupons is

\[
sc(t, t + m) = \sum_{j=1}^{m} D(t, t + j)
\]

The swap fixed coupon rate is obtained by equating the current values of the two coupon streams and is therefore

\[
sc(t, t + m) = \frac{1 - D(t, T)}{\sum_{j=1}^{n} D(t, t + j)}
\]

At inception the value of swap to the fixed rate receiver is

\[
V_r(t) = \sum_{j=1}^{M} sc \ D(t, t + j) + (1 - D(t, T))
\]

where we have abbreviated the symbol for the swap rate to \( sc \) as it is constant once the contract has been set up. We note that the value of this swap will increase if the price of the zero coupon bond rises – that is, if interest rates fall.

A forward swap is a swap which is designed to start at some time in the future. Suppose we consider a forward swap that becomes effective after \( m \) time periods from today and thenceforth has a term of \( n \) further periods. This swap is arranged at current time \( t \). The agreement is to exchange fixed rate interest payments for floating rate payments throughout a future time interval of length \( n \). The coupon rate for the forward swap can be obtained using the same approach as before. Denoting the forward swap rate by
When $m = 0$ we are back to the formula for a plain vanilla swap. The expression for the coupon for the forward swap also does not involve knowledge of the future interest rate movements. Everything that is needed to price the forward swap is known at current time $t$.

A swaption is an option to enter a swap at some future time. Upon maturity of the swaption the owner of the swaption will only exercise it if it is in the money. Suppose the swaption gives its owner the option to enter a receiver swap when the swaption contract matures. This type of swaption is known as a ‘right to receive’ or a receiver swaption. If a firm owns a receiver swaption with a strike price of 7% it will compare the market swap rate with the strike rate upon option maturity. For example if the market swap rate at swaption maturity is 5% then the firm should exercise the swaption because the rate of 7% guaranteed in the strike price provides a better deal than exercising the option. It is preferable to receive fixed rate coupons of 7% than the market rate of 5%. By entering a receiver swaption an institution protects itself against the risk that interest rates will have fallen when the swaption contract matures. This is exactly the type of interest rate risk exposure in the guaranteed annuity option.

It is well known that there is a direct correspondence between swaptions and options on coupon paying bonds. Suppose the strike price for a receiver swaption is $K$. Assume the swaption matures at time $m$ and then it has a tenor of $n$. The payoff to the holder of the swaption at maturity (time $T$) is

$$[X - (sc(T, T + n))^+] \sum_{j=1}^{n} D(T, T + j)$$

where $sc(T, T + n)$ is the swap rate at time $m$ and $D(T, T + j)$ is the price at time $T$ of the pure discount bond which matures at time $(T + j)$. At current time zero these quantities are, in general, random variables. However at time $T$ the prevailing swap rate can be expressed in terms of pure discount bond prices at that time. We have

$$sc(T, T + n) = \frac{1 - D(T, m + n)}{\sum_{j=1}^{n} D(T, m + j)}$$
When we substitute this expression into expression (14) and simplify we find that the payout at the swaption maturity is

$$\max \left[ X \sum_{j=1}^{n} D(T, T + j) + D(T, T + n) - 1, 0 \right]$$

This last expression is the payoff function for a bond option with strike price of unity maturing at time $T$. The underlying bond has a face value of one unit, makes coupon payments of $X$ (the swaption strike price) and has a remaining term to maturity of $n$ periods when the option matures. We can obtained closed form expressions for the value of this option using the same type of assumptions as in section 3.

Pelsser shows how to incorporate mortality risk to replicate the expected payoff under the guaranteed annuity option. He assumes as we do that the mortality risk is independent of the financial risk and that the force of mortality is deterministic. In this way he ends up with an expression for the price of the guaranteed annuity option as portfolio of long dated receiver swaptions.

The advantage of this approach is that the swaptions incorporate the right type of interest rate options. Pelsser calls this approach the static hedge since there is no need for dynamic hedging. This is a great advantage given the problems of hedging long term interest rate options with more basic securities such as bonds and swaps. The swaption solution still has problems in dealing with the stock price risk.

6 Hedging

We now discuss the issues involved in hedging the risk using traded securities. Dynamic hedging long term interest rates derivatives contracts is more challenging than pricing them. There are several reasons for this but one of the main ones is that we need to have a reliable model of interest rate dynamics over the long term in order to set up an effective hedging strategy. It turns out to be much easier to develop a reasonable pricing model. So in the first part of this section we discuss the relationship between pricing and hedging. Then we summarize a number of recent studies which support our thesis. Finally we summarize some of our work on this topic in the context of hedging guaranteed annuity options.

We first discuss the standard no arbitrage pricing model where there is a perfect frictionless market with continuous trading. If the market is
complete then any payoff can be hedged with traded securities. Since there is no arbitrage the current price of the derivative must be equal to the current price of the replicating portfolio. If an institution sells this derivative then it can take the premium (price) and set up the replicating portfolio. As time passes it can dynamically adjust the position so that at maturity the value of the replicating portfolio is exactly equal to the payoff on the derivative. In an ideal world where the model assumptions are fulfilled it should be possible to conduct this replication program without needing any additional funds. The initial price should be exactly enough.

We can use as an example the standard Black Scholes Merton formula for the price of a European call option. In this case the underlying asset is assumed to follow geometric Brownian motion with a constant volatility parameter. The call price formula can be viewed as a portfolio with a long position in the asset and a short position in the risk free bond. To hedge this call option an institution should continuously adjust these positions over the life of the option. If the asset price dynamics correspond to those assume and the other assumptions are realized then the value of the replicating portfolio at option maturity will equal the payoff on the call option.

In the real world the assumptions of these models are never exactly fulfilled. Here are three examples

- The asset price dynamics may be incorrectly specified.
- It is not possible to rebalance the replicating portfolio on a continuous basis. Instead it has to be rebalanced at discrete intervals.
- There are transaction costs on buying and selling the traded assets

The impact of these deviations from the idealized assumptions has been explored in the Black Scholes Merton world. Boyle and Emanuel (1980) showed that if the portfolio is rebalanced at discrete intervals there will be a hedging error which tends to zero as the rebalancing becomes more frequent. In the presence of transaction costs the frequency of rebalancing involves a trade off between the size of the hedging error and the cost of transaction. If the process that generates the market prices deviates from the model on which the option pricing model is based there will be hedging errors. This is because the portfolio weights that would be required to replicate the payoff under the true model will be different from the portfolio weights computed under the assumed model. This point has been explored in the case of stock options

To illustrate how we can calibrate an incorrect model to give the market price of a derivative assume that stock price dynamics follow a process with stochastic volatility. In the case of a given call option we can always find a value of the Black Scholes volatility that reproduces the market price of the call when the true dynamics include stochastic volatility. However as shown by Melino and Turnbull(1995) the use of the simple Black Scholes model, in the presence of stochastic volatility, will lead to large and costly hedging errors especially for long dated options.

In the case of stochastic interest rates the same considerations apply. It is possible to have a simple model that does a reasonable job at pricing interest rate derivatives but yet is inadequate for hedging purposes. Canabarro(1995) uses a two factor simulated economy to show that although ne factor models can produce accurate for interest rate derivatives prices these models lead to poor hedging performance. Gupta and Subrahmanyam (2001) show using actual data that, while a one factor model is adequate for pricing caps and floors, a two factor model performs better in hedging these types of interest rate derivatives. Suppose we use a particular interest rate model: model A for hedging purposes. Model A is used to compute the weights in the replicating portfolio. In general the better Model A reflects the way interest rates actually behave the more effective will be the hedge.

There is currently an ongoing debate concerning the choice of stochastic interest rate models. Litterman and Scheinkman(1991) demonstrated that most of the variation in interest rates could be explained by three stochastic factors. Dai and Singleton(2000) examine three factors models of the so called affine class. They are know as affine models because the short term rate of interest is linear functions of the underlying state variables. The classical Cox Ingersoll Ross model and the Vasicek model are the best known examples of the affine class. These models have the attractive property that bond prices become exponentials of affine functions and are usually easy to evaluate. Dai and Singelton find reasonable empirical support for some versions of the three factor affine model using swap market data for the period April 1987 to August 1996.

In the context of guaranteed annuity options we require an interest rate model that describes interest rate behavior over much longer time span. Ahn, Dittmar and Gallant(2002) recently provide support for quadratic term structure models. They are known as quadratic models because the short term
rate of interest is a quadratic function of the underlying state variables. Their empirical test uses US bond data for the period 1946-1991 and they conclude that the quadratic three factor model

*provides a fairly good description of term structure dynamics and captures these dynamics better than the preferred affine term structure model of Dai and Singleton.*

Bansal and Zhou (2002) show that the affine models are also dominated by their proposed regime switching model. Their empirical test are based on US interest rate data for the period 1964-1995. Even a casual inspection of the data suggests the existence of different regimes. They conclude that their empirical evidence provides considerable support for the regime switching model and that standard models, including the affine models with up to three factors, are sharply rejected by the data. Regime switching models have been extensively used by Hardy (2002) to model equity returns in the context of pricing and risk management of equity indexed annuities.

In view of the correspondence between the interest rate derivatives in guaranteed annuity options it is instructive to examine some recent results on hedging swaptions. This is a topic of current interest and papers by Andersen and Andreasen (2002), Fan, Gupta and Ritchken (2001,2002), Driessen, Klaasen and Melenberg (2002), and Longstaff, Santa-Clara and Schwartz (2001). A main conclusion of these papers that multi-factor models are necessary for good hedging results. The empirical tests in these papers tend to use relatively short observation periods around three to five years. Swaption data is unavailable for long periods since the instruments first were created in the late 1980’s. Hence these models are being tested over the 1995-2000 period when interest rates were fairly stable. If the swaption data were available over longer periods it seems likely that a regime switching rate model would be required to do an adequate hedging job.

### 7 The Equity Risk

We have already shown in equation (1) that the amount of the payoff on the UK guaranteed annuity option is a function of the amount of the maturity proceeds. This amount depends on the stock market performance over the life of the contract. For unit linked policies the maturity amount depends directly on the performance of the underlying fund. In the case of with profits contracts the policy proceeds depend on the bonuses declared by the
insurance company and the size of these bonuses is positively correlated with stock market performance. In general the better the stock market does the larger potential liability under a these options. We refer to this risk as the equity risk. In this section we discuss how the inclusion of this risk impacts the pricing and risk management of the guaranteed annuity options.

We first deal with the pricing issue. We have seen in section 3 that under some strong assumptions about the joint dynamics of interest rates we can obtain simple pricing formula. Specifically if we assume that equity returns are lognormal and that interest rates are governed by a one factor Vasicek model, we can obtain a simple valuation formula for the price of the guaranteed annuity option. Although our formula assumed independence between equity returns and bond dynamics the formula can be modified to handle the case when there is correlation between stocks and bonds. Ballotta and Haberman have also derived a formula under these assumptions. The key insight is that since the equity index and the zero coupon bond are assumed to have bivariate lognormal distribution we can derive a Black Scholes type equation for the current price of the guarantee.

We can illustrate the key issues involved in pricing and hedging when the equity risk is included if we consider a simpler contract. This contract has the following payoff at time $T$

$$S(T) \max(D(T, T + j) - K, 0)$$

It corresponds to an option on the zero coupon bond which matures at time $(T + j)$ and where the payoff is directly related to the value of the reference index. Thus, there is no mortality risk here and we are considering just a single zero coupon bond at maturity rather than a the linear combination of zero coupon bonds we had before. We can derive a simple closed form expression for the price of this contract if assume that under the forward measure $(Q_T)$ the random variables $S(T)$ and $D(t, T + j)$ have a bivariate lognormal distribution with expected values given by

$$E_{Q_T} [\log(D(T, T + j)) | t] = \frac{\sigma_p^2}{2}, \quad E_{Q_T} [\log(S(T)) | t] = \mu_S,$$

and variance-covariance matrix

$$\Sigma = \begin{bmatrix} \sigma_p^2 & \rho \sigma_p \sigma_S \\ \rho \sigma_p \sigma_S & \sigma_S^2 \end{bmatrix}.$$
The option price at time $t$ is given by

$$S(t) \left( \frac{D(t, T + j) e^{\rho \sigma S}}{D(t, T)} N(h_3) - KN(h_4) \right)$$

(15)

where

$$h_3 = \log \frac{D(t, T + j) e^{\rho \sigma S}}{D(t, T) K} + \frac{\sigma_P}{2},$$

$$h_4 = \log \frac{D(t, T + j) e^{\rho \sigma S}}{D(t, T) K} - \frac{\sigma_P}{2}.$$

Formula (15) incorporates both equity risk and interest rate risk. We see that the option price is an increasing function of the correlation coefficient $\rho$. Indeed the price is quite sensitive to the value of $\rho$ and for plausible parameter values the option price for $\rho = 0$ is roughly double that for $\rho = -1$ and half the price corresponding to $\rho = 1$ Correlations are notoriously difficult to forecast and so we conclude that when the equity risk is assumed to be correlated with the interest rate risk, pricing the option becomes more difficult.

It is now well established in the empirical literature that equity prices do not follow a simple lognormal process. There is mounting evidence that some type of stochastic volatility model does a better job of modelling equity returns. Hardy(2003) provides evidence that regime switching model does a good job of fitting the empirical distribution of monthly stock returns. Andersen, Benzoni and Lund(2002) demonstrate that both stochastic volatility and jump components are present in the S and P daily index returns. Several authors\textsuperscript{11} have shown that these models produce significant pricing deviations from the lognormal Black Scholes option prices. Thus it is unlikely that the simple model underlying equation(15) will have the flexibility to price the option accurately.

When we turn to hedging matters become worse. There are two reasons. First we require a good model of the joint dynamics of bonds and equities that will be robust over long time periods. There appears to be no obvious model that would fulfill these requirements. Second even if we are willing to

adopt the pricing model in (15) the resulting hedging implementation leads to some practical problems.

To hedge the option based on this model we need to invest in three assets. The first is an investment in the underlying equity index equal to the current market value of the option. We denote the number of units invested in the index by $H_1(t)$ where

$$H_1(t) = \left( \frac{D(t, T+j) e^{\rho \sigma^2 S}}{D(t, T)} N(h_3) - KN(h_4) \right)$$

The second consists of an investment of $H_2(t)$ units of the zero coupon bond which matures at time $(T + j)$, where

$$H_2(t) = S(t) \frac{e^{\rho \sigma^2}}{D(t, T)} N(h_3)$$

The third consists of an investment of $H_3(t)$ units of the zero coupon bond which matures at time $(T)$, where

$$H_3(t) = -S(t) \frac{D(t, T+j) e^{\rho \sigma^2 S}}{D(t, T)^2} N(h_3)$$

Note that the value of the initial hedge is

$$H_1(t)S(t) + H_2(t)D(t, T+j) + H_3(t)D(t, T),$$

which is also equal to the initial price of the option. The last two terms cancel one another.

Suppose the hedge is to be rebalanced at time $(t+h)$. Just before rebalancing the value of the hedge portfolio is

$$H_1(t)S(t+h) + H_2(t)D(t+h, T+j) + H_3(t)D(t+h, T),$$

where $S(t+h), D(t+h, T+j), D(t+h, T)$ denote the market prices at time $(t+h)$ of the three hedge assets. The new hedging weights $H_i(t+h), i = 1, 2, 3$ are computed based on these new asset prices and the value of the revised hedge is

$$H_1(t+h)S(t+h) + H_2(t+h)D(t+h, T+j) + H_3(t+h)D(t+h, T).$$

Yang develops a similar hedging strategy and Hardy(2003) shows that this strategy leads to hedging errors in the case of real data.
If the value of the hedge portfolio after rebalancing increases funds need to be added. If the value of the hedge portfolio after rebalancing goes down funds can be withdrawn. In an idealized world the hedge would be self financing. However in practice hedging is done discretely, there are transactions costs and the market movements can deviate significantly from the those implied by the model. These slippages can lead to considerable hedging errors.

We saw in connection with the interest rate risk that a policy of buying appropriate portfolios of long dated receiver swaptions would cover this risk. However the presence of equity risk means that the number of swaptions has to be adjusted in line with index movements. During a period of rising equity returns an insurer would have to keep purchasing these swaptions and this would be become very expensive as the swaptions began to move into the money. In these circumstances the liability under the guarantee is open ended. The swaption solution does not deal with the equity risk.

7.1 The Mortality Risk

We noted earlier that there was a dramatic improvement in annuitants mortality over the relevant period. This improvement was not anticipated when the contracts were designed. The effect of this improvement was to increase the value of the interest rate guarantee by raising the threshold interest rate at which the guarantee became effective. The structure of the guaranteed annuity option means that the policyholder’s option is with respect to two random variables: future interest rates and future mortality rates. To isolate the mortality option suppose that all interest rates are deterministic but that future mortality rates are uncertain. In this case the option to convert the maturity proceeds into a life annuity is an option on future mortality rates. If on maturity the mortality rates have improved above the level assumed in the contract the policyholder will obtain a higher annuity under the guarantee. On the other hand life expectancies are lower than those assumed in the contract the guarantee is of no value since policyholder will obtain a higher annuity in the open market. under the guarantee. When interest rates are stochastic the mortality option is affected by the level of the interest rates.

Milevsky and Promsilow(2001) have recently analyzed the twin impacts of stochastic mortality and stochastic interest rates in their discussion of guaranteed purchase rates under variable annuities in the United States. They model the mortality option by considered the traditional actuarial force of mortality as a random variable like the hazard rate in credit risk models.
The expectation of this random variable corresponds to the classical actuarial force of mortality. They show that under some assumptions the mortality option can at least in principle be hedged by the insurance company selling more life insurance. The intuition here is that if people live longer the losses on the option to annuitize will be offset by profits on the life policies sold.

While this is an interesting approach there may be some practical difficulties in implementing it. First it may not be possible to sell the insurance policies and in particular to the same type of policyholders who hold the pension contracts. Second, to implement the mortality hedging strategy the insurer requires a good estimate of the distribution of future mortality. Harking back to the UK case it would have been most unlikely for any insurer in the 1970's to accurately predict variance of the distribution of future mortality rates. If the insurer has a sufficiently accurate estimate of the variance of the the future mortality rates to conduct an effective hedging strategy then it should be able to project future mortality improvements to minimize the mortality risk under the guarantee.

8 Lessons

We have discussed the three major types of risks in the guaranteed annuity option and examined the pricing and the feasibility of hedging the risk under these contracts. In this section we will explore the extent to which the approaches discussed in his paper could or should have been applied. We also suggest that this episode has implications for the eduction and training of the actuarial profession particulary in connection with its exposure to ideas in modern financial economics.

It is worth emphasizing that when these guarantees were being written the UK actuarial profession was still using deterministic methods to value liabilities. In particular valuation and premium calculations were based on a single deterministic interest rate. These methods were enshrined in the educational syllabus and rooted in current practice. Such methods are incapable of dealing adequately with options.

The relevant UK actuarial textbook used at the time, Fisher and Young (1965) in discussing guaranteed annuity options stated:

If, when the maturity date arrives, the guaranteed annuity rate is not as good as the office’s own rates or a better purchase can be made elsewhere the option will not be exercised. The office cannot possibly gain from the
transaction and should, therefore, at least in theory, guarantee only the lowest rate that seem likely in the foreseeable future.

However no guidance was provided as to what level this rate should be. Fisher and Young did suggest that conservative assumptions be used and that allowance should be made for future improvements in mortality.

The option may not be exercised until a future date ranging perhaps from 5 to 50 years hence, and since it will be relatively easy to compare the yield under the option with the then current yields it is likely to be exercised against the office. The mortality and interest rate assumptions should be conservative.

The standard actuarial toolkit in use at the time was incapable of assessing the risks under this type of guarantee. However the guarantees were granted and they gave rise to a serious risk management problem that jeopardized the solvency of a number of UK companies. For many companies, the first time that the guaranteed annuity option for maturing contract became in the money was in October 1993. In December 1993, Equitable Life announced that it would cut the terminal bonuses in the case of policyholders who opted for the guarantee. Basically this meant that the guaranteed annuity option policyholders who exercised their guarantee ended up paying for the guarantee. The affected policyholders argued that Equitable’s action made a mockery of their guarantee. The validity of this controversial approach became the subject of a protracted legal dispute. Eventually, in July 2000 the House of Lords settled the matter. It ruled against the Equitable and decreed that the practice of cutting the terminal bonuses to pay for the guarantee was illegal. Equitable faced an immediate liability of 1.4 billion pounds to cover its current liability for the guaranteed annuity options and in December 2000 was forced to close its doors to new business. The oldest life insurance company in the world was felled by the guaranteed annuity option.

This entire episode should provide salutary lessons for the actuarial profession. It is now clear that the profession could have benefited from greater exposure to the paradigms of modern financial economics. An earlier recognition of the usefulness of stochastic simulation would also have helped in monitoring and managing the exposure under the guarantee.
9 References .. to be added